MT 3700 Differential Equations
Modeling via Systems

I. Recall Models for Exponential and Logistic Growth & Decay

1. Exponential Growth & Decay

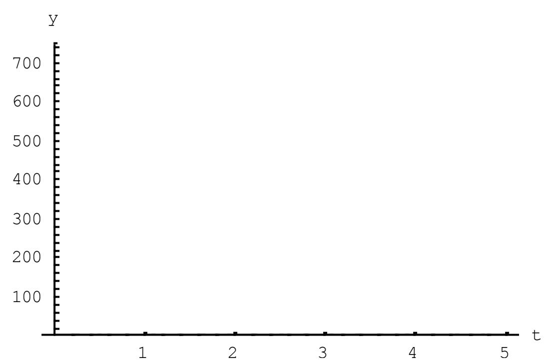
Exponential growth and decay occurs when the *rate of change of the dependent variable is proportional to itself*. That is when $\frac{dy}{dt} = ky$.

a. Find the general solution to the differential equation $\frac{dy}{dt} = ky$

b. What values of k will result in exponential growth? _____________

c. What values of k will result in exponential decay? ______________

d. Sketch the graph of the solution to the exponential growth model $\frac{dy}{dt} = .3y, y(0) = 100$



2. Logistic Growth and Decay

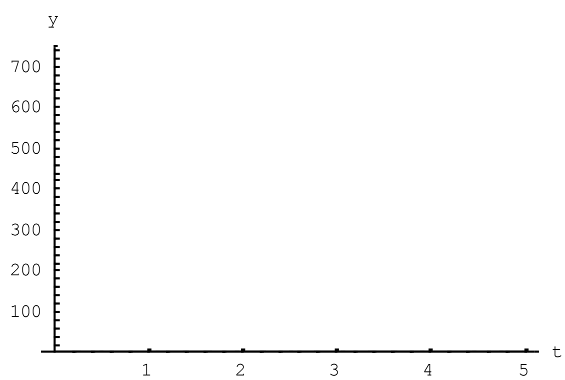
Logistic growth and decay occurs when the *rate of change of the dependent variable is proportional to the product of itself and the room left to grow*.
That is when $\frac{dy}{dt} = ky(1 - \frac{y}{N})$ where *N* represents the maximum value of y that can be supported by the environment.

a. Consider the following Logistic Growth Model: $\frac{dy}{dt} = .3y(1 - \frac{y}{500})$.

b. When y is small (relative to *N=500* , then $\frac{dy}{dt} = .3y(1 - \frac{y}{500})$can be approximated by $\frac{dy}{dt} = .3\,y$ and hence we have __________________________ for small values of y .

c. Eventually y gets closer to *N=500*, and at that point the growth rate $\frac{dy}{dt} = .3y\left(1 - \frac{y}{500}\right)$ gets closer and closer to _________.

d. Sketch the solution to this Logistic Growth Model on the graph below.



II. Revisiting the Predator-Prey Model:

$R(t)$ = population (in hundreds, thousands, millions, etc) of prey at a given time t
$F(t)$ = population of predators at a given time t

Consider, again, the predator-prey system:

$$\frac{dR}{dt} = 2R - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF$$

1. Recall what we mean by a solution to a differential equation, then consider the question: What is a solution to a *system* of differential equations?

2. *Analytical Solutions*: Equilibrium solutions occur when both $\frac{dR}{dt} = 0$ AND $\frac{dF}{dt} = 0$. Find the paired values of R and F that result in equilibrium solutions.

3. *Some more simple analytical solutions*: Consider two extreme initial conditions.

a. How does the system behave in the absence of prey (i.e., when $R(0)=0$)?

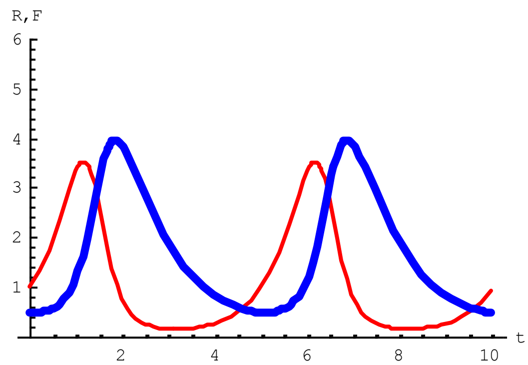
$$\frac{dR}{dt} = 2R - 1.2RF =$$
$$\frac{dF}{dt} = -F + 0.9RF =$$

b. How does the system behave in the absence of predators (i.e., when $F(0)=0$)?

$$\frac{dR}{dt} = 2R - 1.2RF =$$
$$\frac{dF}{dt} = -F + 0.9RF =$$

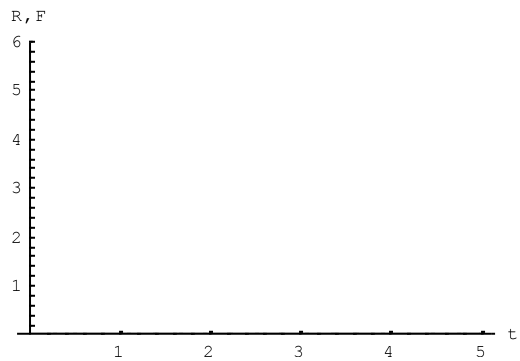
Graphical Techniques:

4. One way to graph the solution of the system is to graph the individual functions $R(t)$ and $F(t)$ belonging to the solution – together on the same set of axes, as shown below, or separately each on its own set of axes. Below are graphs of the solution curves associated with the initial value $(R(0), F(0)) = (1.0, 0.5)$.

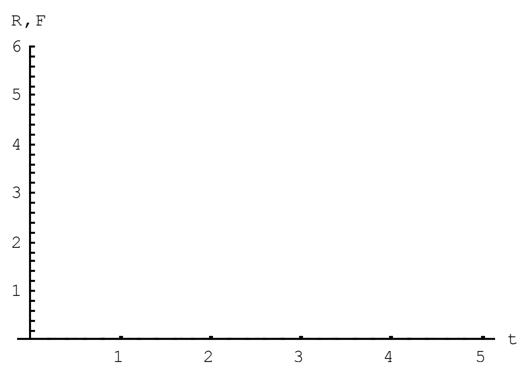


5. Graph the (non-zero) equilibrium solution that we found in part 2.

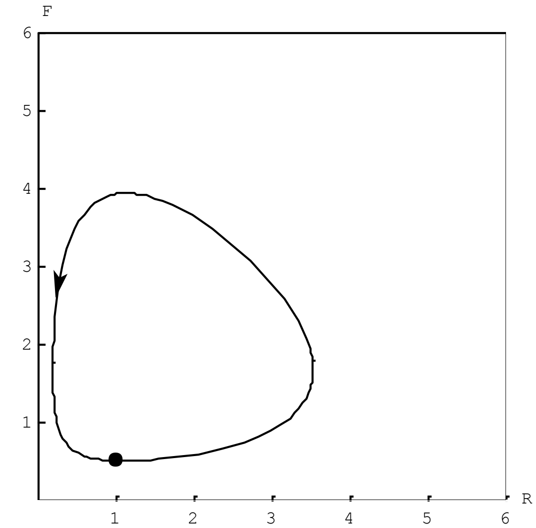
Initial Condition: $(R(0), F(0)) = (\_\_\_\_, \_\_\_\_)$



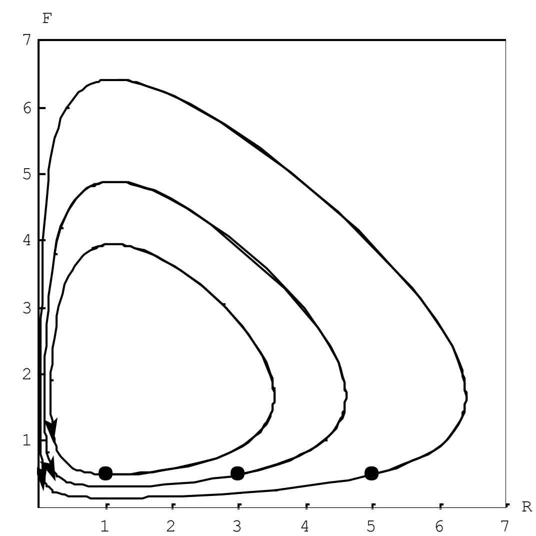
6. Graph the solution we found in part 3(a) satisfying initial condition $(R(0), F(0)) = (0, 5)$



7. **A Phase Plane.** Another way to graph the solution of the system that corresponds to the initial condition $(R(0), F(0)) = (1.0, 0.5)$ is to graph the ordered pairs $(R(t), F(t))$ in the $RF - Plane$



8. We can plot many solution curves to a system on the same phase plane forming a **phase portrait** for our predator-prey system. Graphed below are the solution curves associated with the following initial conditions: $(R(0), F(0)) = (1.0, 0.5)$, $(R(0), F(0)) = (3.0, 0.5)$, and $(R(0), F(0)) = (5.0, 0.5)$



9. Plot the two equilibrium solutions that we found in part 2 on the phase portrait above. Plot the solutions from part 3.

III. A Modified Predator-Prey Model

$$\frac{dR}{dt} = 2R(1-\frac{R}{2}) - 1.2RF$$
$$\frac{dF}{dt} = -F + 0.9RF$$

1. Consider two extreme initial conditions.

   a. How does the system behave in the absence of prey (i.e., when $R(0) = 0$)?

$$\frac{dR}{dt} = 2R(1-\frac{R}{2}) - 1.2RF =$$
$$\frac{dF}{dt} = -F + 0.9RF =$$

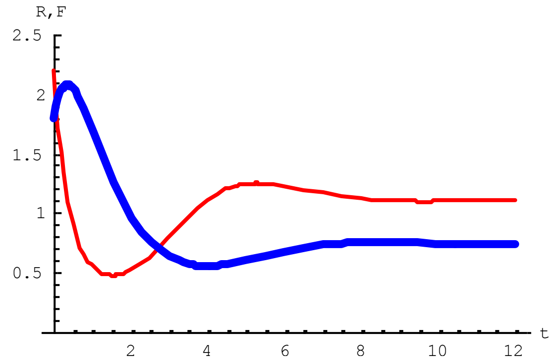
   b. How does the system behave in the absence of predators (i.e., when $F(0) = 0$)?

$$\frac{dR}{dt} = 2R(1-\frac{R}{2}) - 1.2RF =$$
$$\frac{dF}{dt} = -F + 0.9RF =$$

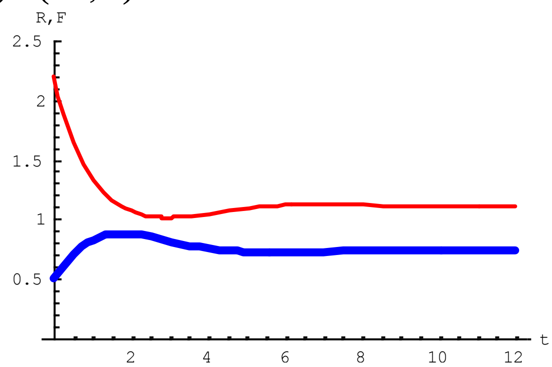
2. Find the equilibrium solutions for this system.

3. Consider the following three solutions to the system for the given initial conditions.

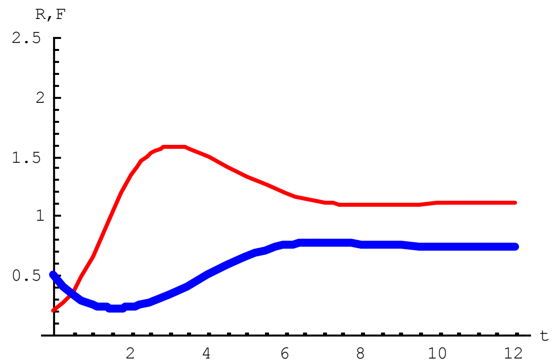
a. $(R(0), F(0)) = (2.2, 1.8)$



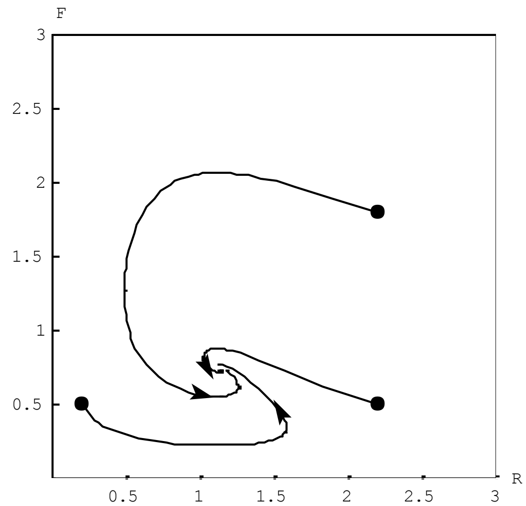
b. $(R(0), F(0)) = (2.2, .5)$



c. $(R(0), F(0)) = (.2, .5)$



4. Below is the phase portrait for these three solutions. Where do all the solutions appear to be heading?



5. Graph the equilibrium solutions that we found in III.2. on the phase portrait above.