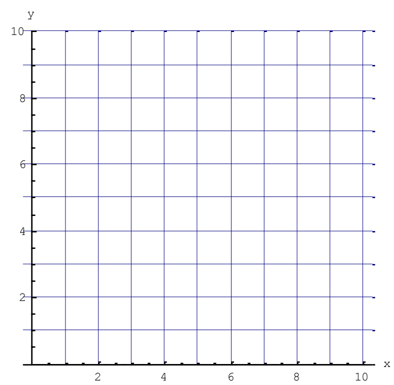
MT 3700 Differential Equations
The Geometry of Systems & Vector Fields

1. Vectors

   a. Definition

   b. Example. Consider the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Graph multiple versions of this vector on the grid below.

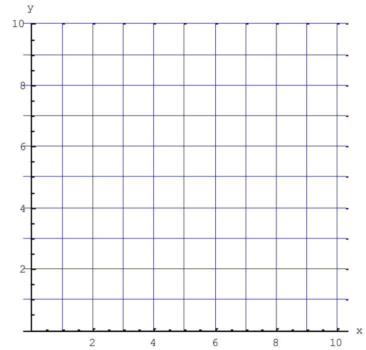


   c. The components of a vector determine both its direction and its magnitude.

   - Slope of $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   - Magnitude of $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

d. Consider the vector $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Graph this vector on the grid below.



- Slope of $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

- Magnitude of $\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

2. Revisiting the Predator-Prey Model: Defining Vector Components

Consider, again, the predator-prey system:

$$\begin{aligned} \frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF \end{aligned} \quad \text{with solution pair } \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$$

Collect these pairs in two vectors:

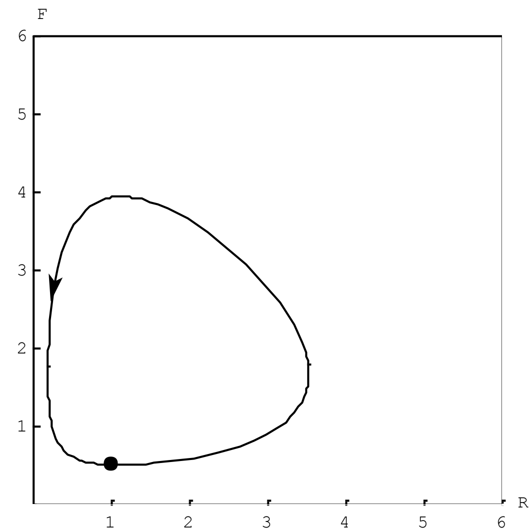
$$\frac{d\vec{P}}{dt} = \qquad\qquad\qquad \text{with} \quad \vec{P}(t) =$$

Rename the DE vector:

$$\frac{d\vec{P}}{dt} =$$

3. Revisiting the Predator-Prey Model: Applying Vector Notation

a. Recall that we looked at the solution curve in the phase-plane satisfying the initial condition, $(R(0), F(0)) = (1.0, 0.5)$:



b. On the phase plane above, identify the vector $\vec{P}(0)$. What does this vector point to?

c. Notice that $\frac{dR}{dt}$ and $\frac{dF}{dt}$ are given by our system of differential equations. So:

$$\frac{d\vec{P}}{dt} = \left(\qquad\qquad\qquad\qquad \right)$$

d. The right hand side of the equality above is a special type of function called a vector-field. A vector-field is any function that takes a vector into a vector. In the case above, for each value of the pair/vector (R,F), a vector is assigned defined by:

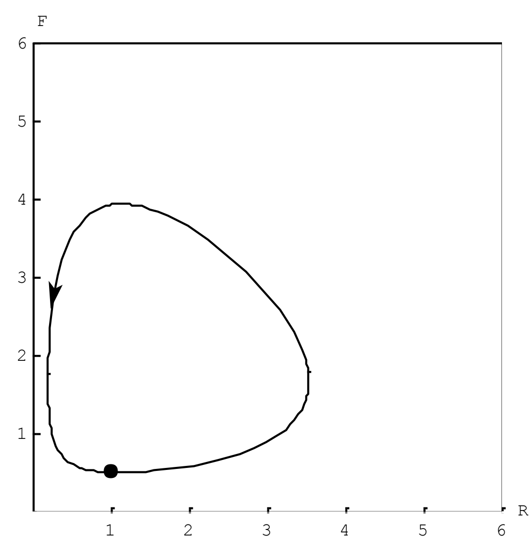
$$\vec{V}\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} 2R-1.2RF \\ -F+0.9RF \end{pmatrix}$$

e. For example, what is the vector assigned by this vector-field to the initial value $(R,F)=(1.0,0.5)$?

$$\vec{V}\begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

f. Recall from Calculus III (some of you), what $\| \frac{d\vec{P}}{dt} \|$ represents geometrically:

g. Draw the vector $\frac{d\vec{P}}{dt}\ (at\ \ t=0)=\vec{V}\begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix}$ in the phase-plane below. Draw it so that its tail is at (1.0,0.5).



4. We can now use a very compact notation for the system of differential Equations:

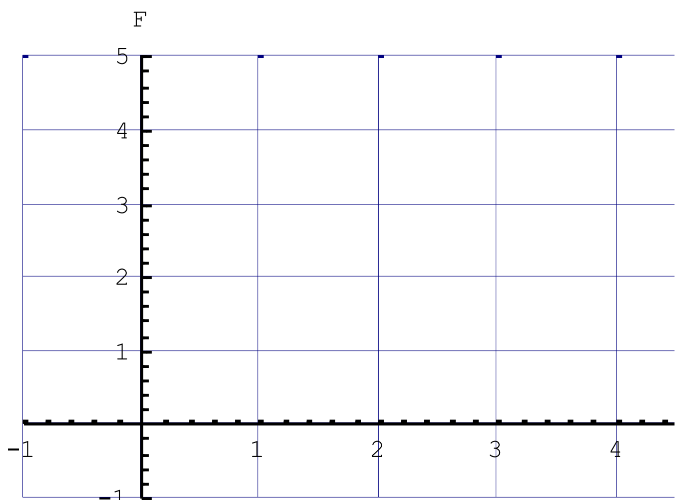
$$\frac{d\vec{P}}{dt} = \vec{V}(\vec{P}),$$

where $\vec{V}$ is the vector field describing the predator-prey system:

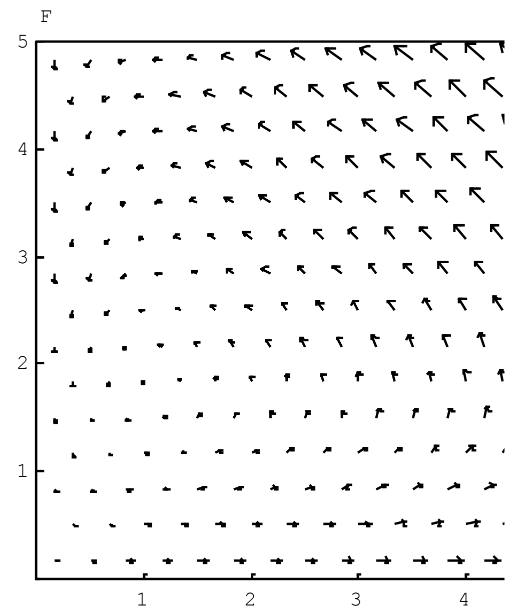
a. How do we graph a vector field? The graph of a vector field is called a vector field diagram.

b. On the blank axes below, graph part of the vector field diagram for $\vec{V}$ by graphing the output vector which $\vec{V}$ associates with each of the following input vectors:

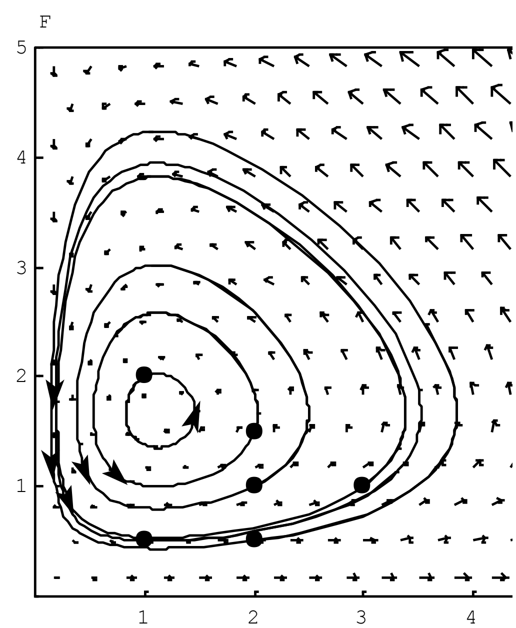
$$\begin{pmatrix} R \\ F \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



c. The complete direction field for $\vec{V}$ is shown below. Sketch in several solution curves corresponding to different initial conditions.

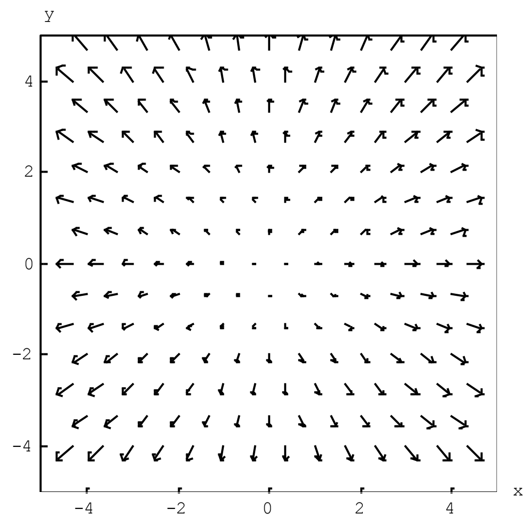


d. Here is the direction field with several solution curves (corresponding to our six initial conditions) drawn.

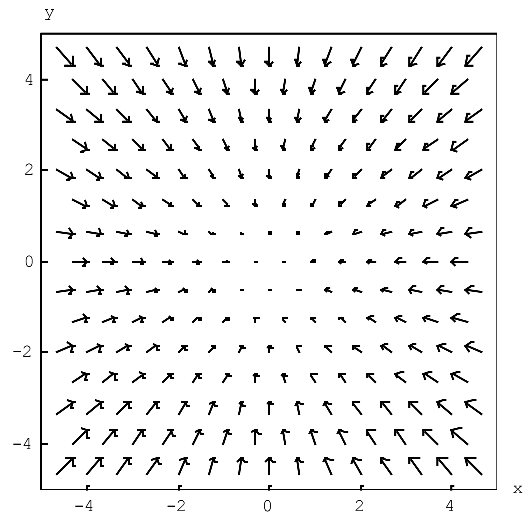


5. Examples of other Vector Fields. Verify these by using *Mathematica.*

a. Example 1: $\vec{F}(x,y)=\begin{pmatrix} x \\ y \end{pmatrix}=(x,y)$



b. Example 2: $\vec{F}(x,y) = \begin{pmatrix} -x \\ -y \end{pmatrix} = (-x, -y)$



c. Example 3: $\vec{F}(x,y) = \begin{pmatrix} -x \\ -2y \end{pmatrix} = (-x, -2y)$