MT 3700 Differential Equations
Bifurcations

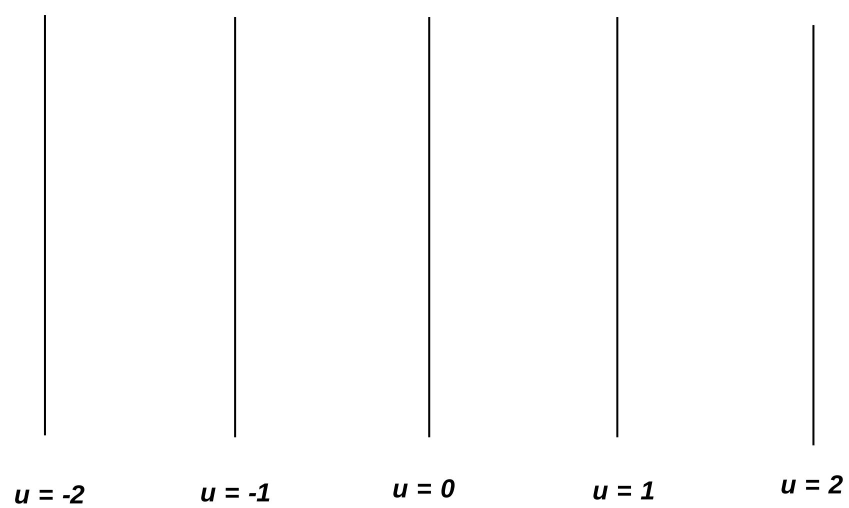
Name ____________________________________

- Recall: We say that a differential equation defined with one or more parameters has a *bifurcation* if a *small change* in the value of the parameter results in a *qualitative change* in the behavior of the solutions to the differential equation.

1. Consider the following one-parameter family of differential equations: $f_\mu(y) = y^3 - \mu y$.

   a. In what ways might phase lines for a family of differential equations show a qualitative change in the behavior of the solutions?

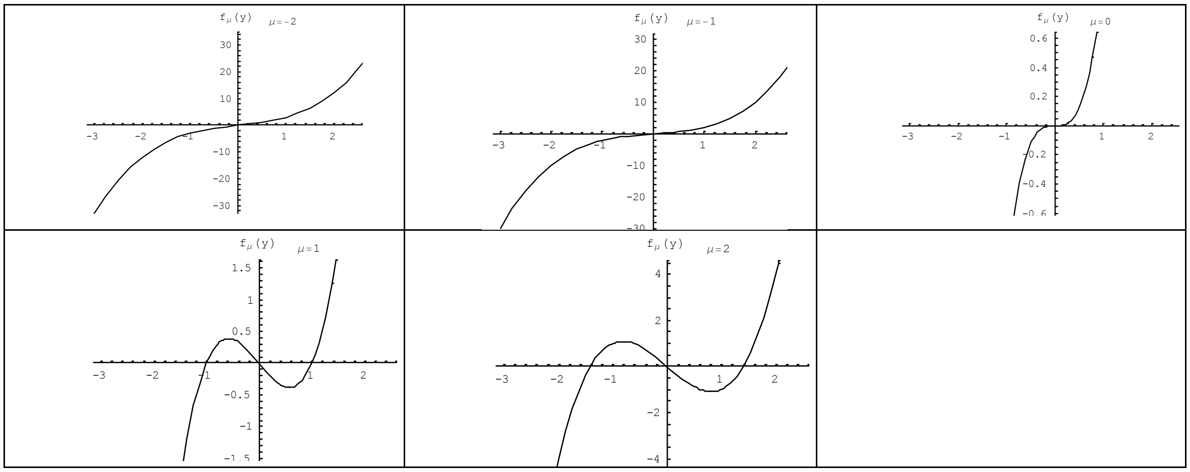
   b. Sketch phase lines for this family of differential equations for $\mu = -2, -1, 0, 1, 2$.



   c. Based on the phase lines above, does this family of differential equations have any bifurcations? If so, where do you think they occur?

   d. Describe the qualitative change in the solutions that occurs at this or these bifurcation value(s).

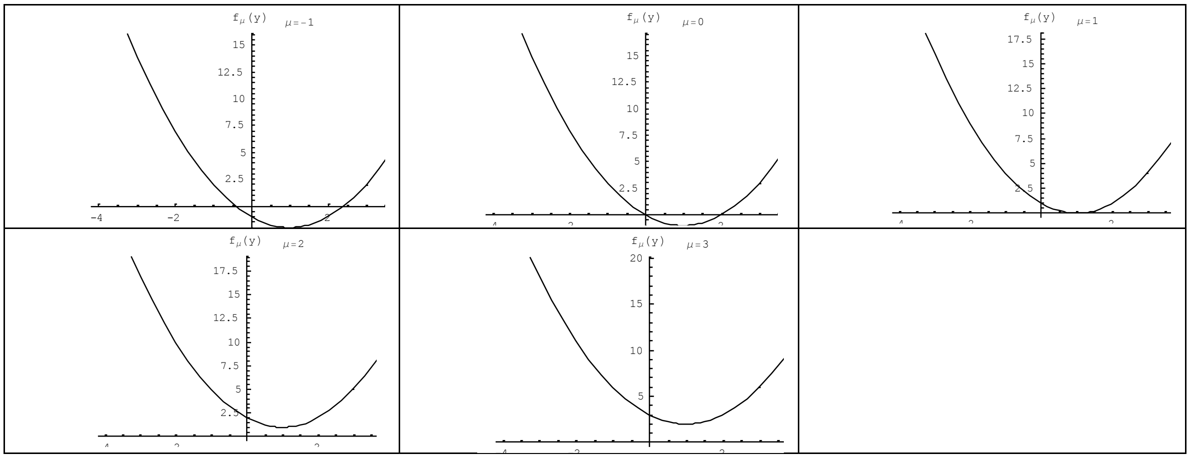
e. Try looking at the following graphs of $f_\mu(y) = y^3 - \mu y$. Can you locate any bifurcation values more accurately given this graphical information?



f. What's missing in the above graphs is accurate information about $f_\mu\,'(y)$ at equilibrium points. Consider the following additional information about $f_\mu(y) = y^3 - \mu y$. Now can you accurately locate any bifurcation values?

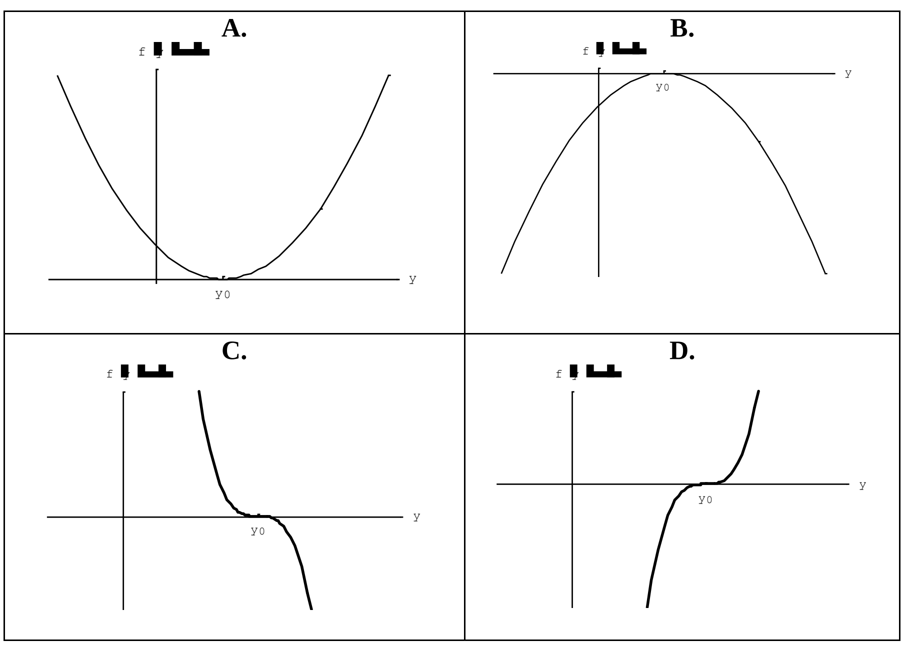
- $f_{-2}\,'(0) = 2$
- $f_{-1}\,'(0) = 1$
- $f_0\,'(0) = 0$
- $f_1\,'(0) = -1$
- $f_2\,'(0) = -2$

2. Let's try another example: $f_\mu(y) = y^2 - 2y + \mu$. We've already looked at the phase lines for this family of differential equations. Consider the graphs of $f_\mu(y) = y^2 - 2y + \mu$ shown below. Can you locate any bifurcation values?



3. Notice that you located the bifurcation values in problems 1 and 2 by looking graphically (in some different ways) at various specified values of the parameter μ. And even in these cases, you were not always able to locate exactly where a bifurcation occurred. What we need is a method (other than "try lots of values for μ ") for locating the bifurcation values *exactly* – an algebraic method:

   We can see that from these examples and from earlier examples in class, that given a one-parameter family of differential equations $\frac{dy}{dt} = f_\mu(y)$, bifurcation values occur at values of $\mu = \mu_0$ for which $f_{\mu_0}(y_0) = 0$ (we can spot characteristics of a bifurcation value at equilibrium points) AND where $f'_{\mu_0}(y_0) = 0$. Pictured below are four graphs that meet both of these conditions at point $y = y_0$.



Which of the following graphs of $\frac{dy}{dt} = f_\mu(y)$ *might* result in a bifurcation value at $y = y_0$?

4. So, now we have the tools to create a METHOD for locating bifurcation values. We will need to look for values of μ that have the following characteristics at some point $y = y_0$.

   ✓ $f_\mu(y_0) = 0$
   ✓ $f_\mu^{'}(y_0) = 0$

5. Let's build the METHOD for locating bifurcation values around the following example. Suppose we want to find the bifurcation values for $\frac{dy}{dt} = f_\mu(y) = y(1-y)^2 + \mu$.

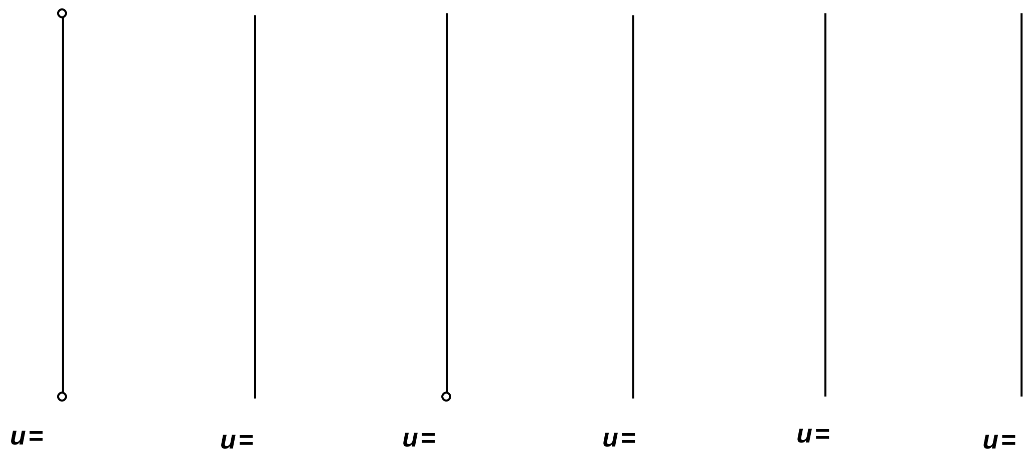
a. STEP 1: Calculate $f_\mu^{'}(y)$. (We start here because often times taking the derivative will eliminate the parameter.)

b. STEP 2: Look for values of $y = y_0$ that cause $f_\mu^{'}(y_0) = 0$. These y-values should help us locate the μ -values.

c. STEP 3: Use the y_0 values found in STEP 2 together with the condition that $f_\mu(y_0) = 0$ to find the μ value associated with each y_0 value. These are your potential bifurcation values.

Bifurcation values occur when μ = ______________________________

d. Use phase lines to sketch solutions to this differential equation for $\mu - values$ in the vicinity of your bifurcation values.



e. Open a *Mathematica* notebook and graph $f_{\mu}(y)$ for each value of μ that you selected for your phase lines. Study carefully what is happening as you approach the bifurcation values.