MT 1800 Calculus I
Worksheet 2.2 – The Derivative at a Point

Purpose: To review average and instantaneous rates of change.

1. Recall that the average rate of change of a function $f(x)$ over an interval $[a,b]$ can be represented geometrically by the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$, that is:

$$\text{Avg. rate of change of } f(x) \text{ on } [a,b] = \frac{f(b)-f(a)}{b-a}.$$

2. The instantaneous rate of change of $f(x)$ at $x = a$ corresponds geometrically to the slope of the tangent to the graph of $f(x)$ at the point $(a, f(a))$.

   The instantaneous rate of change of $f(x)$ at $x = a$ is the derivative of f at $x = a$ and is written $f'(a)$.

   To find $f'(a)$, we compute

$$lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

   and declare the derivative $f'(a)$ to be the value of the limit, provided this limit exists.

   Note that h could be positive or negative and this is equivalent to approaching a from the right and the left.

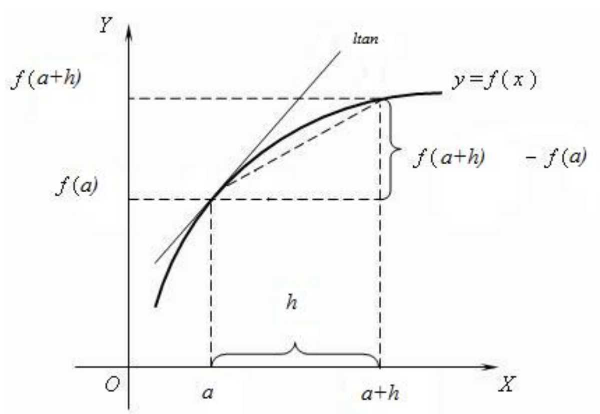
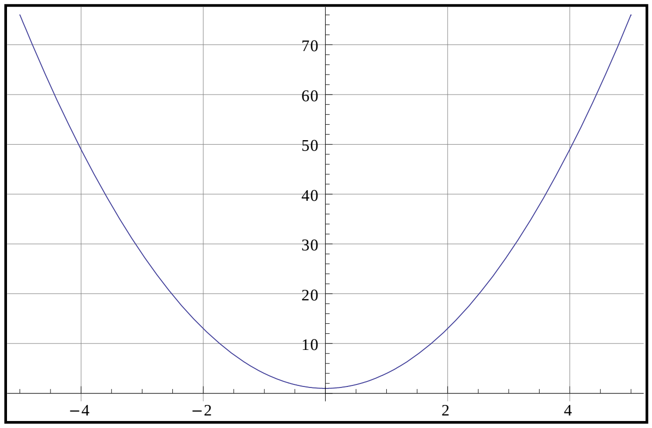


Fig. 1

The purpose of this worksheet is to estimate and calculate rates of change and derivatives for several different functions. We will see that this can be done numerically, graphically, and algebraically. By algebraically, we mean using the limit definition given above.

A. Define a function:

$f(x) = 3x^2 + 1$



Plot[3x^2 + 1, {x, −5,5}, GridLines → Automatic]

- We select two points of interest :

$(x_{1,}y_1) = (0, f(0)) = (0,$ ______$)$

$(x_{2,}y_2) = (4, f(4)) = (4,$ ______$)$

- Calculate the average rate of change of $f(x)$ over the interval [0,4]

Average Rate of Change = $\dfrac{\Delta f}{\Delta x}$ = ______________________________

- Graph the secant line connecting these two points.
- Find the equation of this secant line.

s(x) = ____

- Can we compare without making any calculations? Think geometrically and look at the graph!

Average rate of change of $f(x)$ over [0,4] __________Average rate of change of $f(x)$ over [0,2]

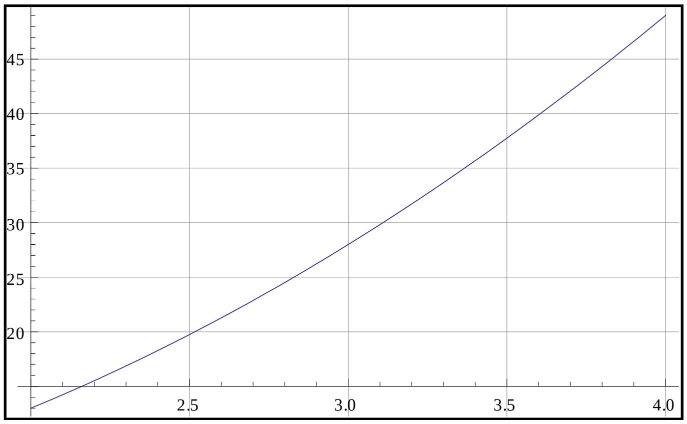
Average rate of change of $f(x)$ over [0,4] __________Average rate of change of $f(x)$ over [2,4]

Average rate of change of $f(x)$ over [-2,0] __________Average rate of change of $f(x)$ over [-4,0]

Average rate of change of $f(x)$ over [-4,4] = _________

Graphically approximate the Instantaneous Rate of Change of $f(x) = 3x^2 + 1$ at x = 3

Let's zoom in around 3 in the picture above:



Plot[$3x$^2 + 1, {x, 2,4}, GridLines → Automatic]

Draw the tangent line to the graph of $f(x) = 3x^2 + 1$ at $x = 3$.

Approximate the slope of the tangent? _________

Numerically find the Instantaneous Rate of Change of $f(x) = 3x^2 + 1$ at x = 3

Recall that the instantaneous rate of change of $f(x)$ at $x = 3$ is the derivative of f at $x = 3$ and is written_______________.

To find ______, we compute_____________________

and declare the derivative _________to be the value of the limit, provided this limit exists.

Because we need to check if the limit exists we will compute the limits from the right and the left and then compare them.

Use the table below to find $lim_{h \to 0^+} \frac{f(3+h)-f(3)}{(3+h)-3}$

a	$a+h$	$f(a)$	$f(a+h)$	$\frac{f(a+h)-f(a)}{(a+h)-a}$
3	3.01			
3	3.001			
3	3.0001			
	↓			↓
	3^+			

Now find $lim_{h\to 0^-}\frac{f(3+h)-f(3)}{(3+h)-3}$

a	$a+h$	$f(a)$	$f(a+h)$	$\frac{f(a+h)-f(a)}{(a+h)-a}$
3	2.99			
3	2.999			
3	2.9999			
	↓			↓
	3^-			

We conclude that

Algebraically find the Instantaneous Rate of Change of $f(x) = 3x^2 + 1$ **at** x **= 3**

$$f'(a) = lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Find the equation of the tangent line to the graph of $f(x) = 3x^2 + 1$ at $x = 3$.

B. Define a function:

$f(x) = \sqrt{x}$

Graphically approximate the Instantaneous Rate of Change of $f(x) = \sqrt{x}$ at x = 9

a. Draw the graph of the function in Mathematica. Print the graph and draw a tangent line by hand at $x = 9$.

b. Approximate the slope of the tangent to the graph of the function $f(x) = \sqrt{x}$ at $x = 9$

Numerically find the Instantaneous Rate of Change of $f(x) = \sqrt{x}$ at x = 9

c. Recall that the instantaneous rate of change of $f(x)$ at $x = 9$ is the derivative of f at $x = 9$ and is written________________.

To find ______, we compute______________________

and declare the derivative _________to be the value of the limit, provided this limit exists.

Because we need to check if the limit exists we will compute the limits from the right and the left and then compare them.

d. Use the table below to find $lim_{h\to 0^+} \frac{f(9+h)-f(9)}{(9+h)-9}$

a	$a+h$	$f(a)$	$f(a+h)$	$\frac{f(a+h)-f(a)}{(a+h)-a}$
9	9.01			
9	9.001			
9	9.0001			
	↓			↓
	9^+			

e. Now find $lim_{h\to0^-}\frac{f(9+h)-f(9)}{(9+h)-9}$

a	$a+h$	$f(a)$	$f(a+h)$	$\frac{f(a+h)-f(a)}{(a+h)-a}$
9	8.99			
9	8.999			
9	8.9999			
	↓			↓
	9^-			

f. We conclude that the derivative of $f(x)=\sqrt{x}$ at x = 9

Algebraically find the Instantaneous Rate of Change of $f(x)=\sqrt{x}$ at x = 9

$$f'(a) = lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$

g. $f'(9) =$

h. Find the equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at x = 9 .