MT1800 – Calculus 1

Project 1 = Take home part for test 1

Instructions:

Work on the following 5 problems individually. You can consult your book and class notes. Problems 2 and 3 include all the out-of-class background you need to answer the questions.

Each problem is worth 8 points, for a total of 40% of your grade in Test 1.

To receive full credit you are required to:

- → *Show all your work in detail,*
- → *provide a print out of your Mathematica notebook with the requested graphs,*
- → *use correct mathematical notation,*
- → *write full sentences stating your answers clearly, and*
- → *turn in the project on Monday, September 15th, 2008, during your class time.*

Problems 2, 3, and 4 come from Calculus for Biology and Medicine by Claudia Neuhauser.

Problems:

1. **Problem 31 p. 8 in your book (linear function)**

2. **Monod Growth Function (rational function)**

The following function is frequently used to describe the per capita growth rate of organisms when the growth rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations. We denote the concentration of the nutrient by N; the per capita growth rate $r(N)$ is given by the Monod growth function

$$r(N) = \frac{aN}{k+N}, N \geq 0$$

where a and k are positive constants. The graph of $r(N)$ is a piece of a hyperbola. It shows a decelerating rise approaching the saturation level a, which is the maximal specific growth rate. When $N = k$, then $r(N) = a/2$; for this reason, k is called the half saturation constant. The growth rate increases with nutrient concentration N; however, doubling the nutrient concentration has a much bigger effect on the growth rate for small values of N than when N is already large. This type of function is also used in biochemistry to describe enzymatic reactions; it is then called the Michaelis-Menten function.

Investigate the Monod growth function.

a) Graph $r(N)$ for:
   i. $a = 5 \text{ } and \text{ } k = 1$
   ii. $a = 5 \text{ } and \text{ } k = 3$
   iii. $a = 8 \text{ } and \text{ } k = 1$

   Place all three graphs in one coordinate system.

b) Based on the graphs in a) describe in words what happens when you change a. What does this mean in terms of the concentration of the nutrient?
c) Based on the graphs in a) describe in words what happens when you change k. What does this mean in terms of the concentration of the nutrient?
d) What happens to $r(N)$ as N increases? Use this to explain in words why a is called the saturation level. What does this mean in this context?
e) Show that k is the half-saturation constant; that is, show that if $N = k$, then $r(N) = a/2$.

Assume now that

$$\boldsymbol{r(N) = \frac{5N}{k+N}, N \geq 0}$$

f) Find the percentage increase when the nutrient concentration is doubled from $N = 0.1$ to $N = 0.2$.

g) Find the percentage increase when the nutrient concentration is doubled from $N = 10$ to $N = 20$.

h) Compare your findings when you in f) and g). What is happening? What does this mean in terms of the concentration of the nutrient? (This is an example of diminishing return).

3. A Chemical reaction (polynomial function)

Consider the reaction rate of the chemical reaction

$$A + B \rightarrow AB$$

in which the molecular reactants A and B form the molecular product AB. The rate at which this reaction proceeds depends on how often A and B molecules collide. The law of mass action states that the rate at which this reaction proceeds is proportional to the product of the respective concentrations of the reactants. Here, concentration means the number of molecules per fixed volume. If we denote the reaction rate by R and the concentration rate of A and B by $[A]$ and $[B]$, respectively, then the law of mass action says that

$$R\alpha[A] \cdot [B].$$

Introducing the proportionality factor k, we can rephrase the statement above as follows:

$$R = k[A] \cdot [B]$$

Note that $k > 0$ because $[A], [B]$, and R are positive. We assume now that the reaction occurs in a closed vessel; that is, we add specific amounts of the reactants A and B to the vessel at the beginning of the reaction and then let the reaction proceed without further additions.

We can express the concentrations of the reactants A and B during the reaction in terms of their initial concentrations a and b and the concentration of the molecular product $[AB]$.

If $x = [AB]$, then

$$[A] = a - x \text{ for } 0 \leq x \leq a \text{ and } [B] = b - x \text{ for } 0 \leq x \leq b \ .$$

The concentration of AB cannot exceed either of the concentrations of A or B. Therefore we get

$$R(x) = k(a-x)(b-x) \text{ for } 0 \leq x \leq a \text{ and } 0 \leq x \leq b$$

The condition “$0 \leq x \leq a$ and $0 \leq x \leq b$ “ can be written as $0 \leq x \leq \min\{a, b\}$, where $\min\{a, b\}$ denotes the minimum of a and b. To see that $R(x)$ is indeed a polynomial function, we expand the expression for $R(x)$ as follows

$$R(x) = k(ab - ax - bx + x^2) = kx^2 - k(a+b)x + kab, \quad \text{for } 0 \leq x \leq \min\{a, b\}.$$

Investigate $R(x)$

Assume that initially only A and B are in the reaction vessel and that the initial concentrations are:

$$a = [A] = 3 \ \ and \ b = [B] = 4$$

a) Suppose the reaction rate $R(x)$ is equal to 9 when the concentration of AB is $x = 1$. Use this to find an expression for the reaction rate $R(x)$.
b) Determine the appropriate domain for $R(x)$ and sketch the graph of $R(x)$.

Autocatalytic reaction

An autocatalytic reaction uses its resulting product for the formation of a new product, as in the reaction $A + X \rightarrow X$.

If we assume this reaction occurs in a closed vessel, then the reaction rate is given by

$$R(X) = kx(a - x) \ for \ 0 \leq x \leq a,$$

Where a is the initial concentration of A and x is the initial concentration of X.

c) Show that $R(x)$ is a polynomial and determine its degree.
d) Graph $R(x)$ for $k = 2$ and $a = 6$. Find the value of x at which the reaction rate is maximal.

4. Fish growth (using exponential functions)

Fish are indeterminate growers: that is, they grow through their lifetime. The growth can be described by the von Bertalanffy function

$$L(x) = L_\infty\left(1 - e^{-kx}\right)$$

For $x \geq 0$, where $L(x)$ is the length at age x, and k and L_∞ are positive constants.

a) Graph $L(x)$ for $L_\infty = 20$, for i) $k = 1$) and ii) $k = 0.1$.
b) For $k = 1$, find x so that the length is 90% of L_∞. Repeat for 99% of L_∞. Can the fish ever attain length L_∞? Interpret the meaning of L_∞.
c) Compare the graphs obtained in a). Which growth curve reaches 90% of L_∞ faster?

d) Can you explain what happens to the curve $L(x)$ when you vary k (for fixed L_∞)? Explain what this means in the context of fish growth.

5. Problem 48 p. 60 in your book (trigonometric function)