Section 13.3: The Dot Product

Now we learn a way to multiply two vectors together—sort of. The dot product, which is given by two different but equivalent definitions, will be very useful to us. Suppose you have vectors $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Geometric definition:

$\vec{v} \cdot \vec{w} =$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

Algebraic definition:

$\vec{v} \cdot \vec{w} =$

Note that in both definitions, the dot product is a number, not a vector.

The book uses the law of cosines to prove that the two definitions are equivalent. Let's just check this for one example.

Example: Let $\vec{v} = \vec{i}$ and let $\vec{w} = -\vec{i} + \vec{j}$.

Page 666 of the book gives a few important properties of the dot product. Let's look at a few of them:

Order doesn't matter (that is, the dot product is commutative).

Two nonzero vectors are perpendicular if and only if their dot product is zero. This follows from the fact that $\cos(90) = 0$.

We can use the dot product to compute the length of a vector. Suppose $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Then

The dot product can also be used to find equations of planes.

Example 4 (p. 668): Find the equation of the plane which passes through the point $(1,0,4)$ and is normal (perpendicular) to the vector $-\vec{i}+3\vec{j}+2\vec{k}$.

We want all points (x,y,z) such that the vector $(x-1)\vec{i}+(y-0)\vec{j}+(z-4)\vec{k}$ is perpendicular to $-\vec{i}+3\vec{j}+2\vec{k}$. Now use the dot product:

Now look carefully at the equation of the plane. You can still see that $-\vec{i}+3\vec{j}+2\vec{k}$ is the normal vector! Also, by looking at the first version of the equation, you can see the normal vector and the point $(1,0,4)$. Thus we have a general formula for a plane:

Projections:

Suppose you have a unit vector $\vec{u}$ and some other vector $\vec{v}$. Sometimes it is useful to break $\vec{v}$ into components that are parallel and perpendicular to $\vec{u}$. Let's figure out how to do this.

Example: Find a unit vector $\vec{u}$ in the direction of $3\vec{i}+\vec{j}$. Then find the projection of $\vec{v}=\vec{i}+2\vec{j}$ onto $\vec{u}$.

Example 7 (p. 669): If a wind approaches a sailboat at an angle of 30 degrees from the direction the sailboat is going, how much of this wind can contribute to the motion of the sailboat?