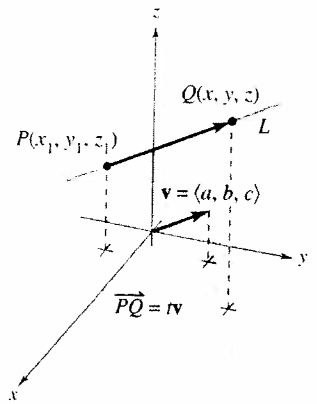
## Section 10.5 Lines and Planes in Space

- Write a set of parametric equations for a line in space.
- Write a linear equation to represent a plane in space.
- Sketch the plane given by a linear equation.
- Find the distance between points, planes, and lines in space.

### Lines in Space




Line $L$ and its direction vector $\mathbf{v}$
**Figure 10.43**

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 10.43, consider the line $L$ through the point $P(x_1, y_1, z_1)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. The vector $\mathbf{v}$ is the **direction vector** for the line $L$, and $a$, $b$, and $c$ are the **direction numbers.** One way of describing the line $L$ is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where $t$ is scalar (a real number).

$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle = \langle at, bt, ct \rangle = t\mathbf{v}$$

By equating corresponding components, you can obtain the **parametric equations of** a line in space.

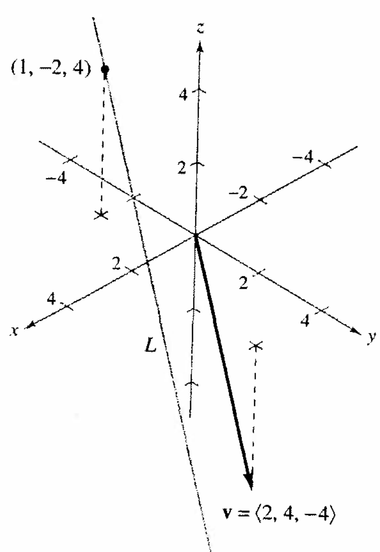
> **THEOREM 10.11 Parametric Equations of a Line in Space**
>
> A line $L$ parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**
>
> $$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

If the direction numbers $a$, $b$, and $c$ are all nonzero, you can eliminate the parameter $t$ to obtain the **symmetric equations** of a line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{Symmetric equations}$$

### *Example 1* Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line $L$ that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4 \rangle$.




The vector $\mathbf{v}$ is parallel to the line $L$.
**Figure 10.44**

**Solution** To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$ (see Figure 10.44).

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t \qquad \text{Parametric equations}$$

Because $a$, $b$, and $c$ are all nonzero, a set of symmetric equations is

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}. \qquad \text{Symmetric equations}$$

Neither the parametric equations nor the symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ would produce the different parametric equations

$$x = 3 + 2t, \quad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

Find a set of parametric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

Begin by using the points $P(-2, 1, 0)$ and $Q(1, 3, 5)$ to find a direction vector for the line passing through $P$ and $Q$, given by

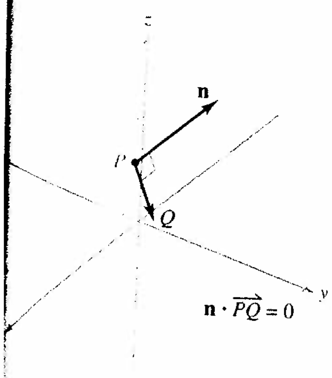
$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle = \langle a, b, c \rangle.$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$ with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t.$$

NOTE As $t$ varies over all real numbers, the parametric equations in Example 2 determine the points $(x, y, z)$ on the line. In particular, note that $t = 0$ and $t = 1$ give the original points $(-2, 1, 0)$ and $(1, 3, 5)$.

## Planes in Space




e normal vector $\mathbf{n}$ is orthogonal to each tor $\overrightarrow{PQ}$ in the plane.
**ure 10.45**

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to it.

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$, as shown in Figure 10.45. This plane consists of all points $Q(x, y, z)$ for which vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$
$$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

The third equation of the plane is said to be in **standard form.**

The plane containing the point $(x_1, y_1, z_1)$ and having a normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented, in **standard form,** by the equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

By regrouping terms, you obtain the **general form** of the equation of a plane in space.

$$ax + by + cz + d = 0 \qquad \text{General form of equation of plane}$$

**In Exercises 13 and 14, find a set of parametric equations of the line.**

**13.** The line passes through the point $(2, 3, 4)$ and is parallel to the $xz$-plane and the $yz$-plane.

**14.** The line passes through the point $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$.

**In Exercises 15 and 16, determine which points lie on the line $L$.**

**15.** The line $L$ passes through the point $(-2, 3, 1)$ and is parallel to the vector $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$.

(a) $(2, 3, 0)$ (b) $(-6, 3, 2)$ (c) $(2, 1, 0)$ (d) $(6, 3, -2)$

**16.** The line $L$ passes through the points $(2, 0, -3)$ and $(4, 2, -2)$.

(a) $(4, 1, -2)$ (b) $\left(\frac{5}{2}, \frac{1}{2}, -\frac{11}{4}\right)$ (c) $(-1, -3, -4)$

**In Exercises 17 and 18, determine if any of the lines are parallel or identical.**

**17.** $L_1$: $x = 6 - 3t,\ y = -2 + 2t,\ z = 5 + 4t$

$L_2$: $x = 6t,\ y = 2 - 4t,\ z = 13 - 8t$

$L_3$: $x = 10 - 6t,\ y = 3 + 4t,\ z = 7 + 8t$

$L_4$: $x = -4 + 6t,\ y = 3 + 4t,\ z = 5 - 6t$

**18.** $L_1$: $\dfrac{x - 8}{4} = \dfrac{y + 5}{-2} = \dfrac{z + 9}{3}$

$L_2$: $\dfrac{x + 7}{2} = \dfrac{y - 4}{1} = \dfrac{z + 6}{5}$

$L_3$: $\dfrac{x + 4}{-8} = \dfrac{y - 1}{4} = \dfrac{z + 18}{-6}$

$L_4$: $\dfrac{x - 2}{-2} = \dfrac{y + 3}{1} = \dfrac{z - 4}{1.5}$

**In Exercises 19–22, determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.**

**19.** $x = 4t + 2,\ y = 3,\ z = -t + 1$

$x = 2s + 2,\ y = 2s + 3,\ z = s + 1$

**20.** $x = -3t + 1,\ y = 4t + 1,\ z = 2t + 4$

$x = 3s + 1,\ y = 2s + 4,\ z = -s + 1$

**21.** $\dfrac{x}{3} = \dfrac{y - 2}{-1} = z + 1, \quad \dfrac{x - 1}{4} = y + 2 = \dfrac{z + 3}{-3}$

**22.** $\dfrac{x - 2}{-3} = \dfrac{y - 2}{6} = z - 3, \quad \dfrac{x - 3}{2} = y + 5 = \dfrac{z + 2}{4}$

**In Exercises 23 and 24, use a computer algebra system to graph the pair of intersecting lines and find the point of intersection.**

**23.** $x = 2t + 3,\ y = 5t - 2,\ z = -t + 1$

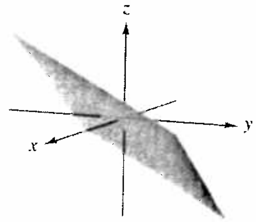
$x = -2s + 7,\ y = s + 8,\ z = 2s - 1$

**24.** $x = 2t - 1,\ y = -4t + 10,\ z = t$

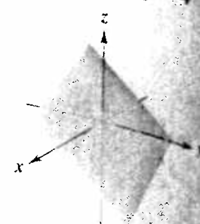
$x = -5s - 12,\ y = 3s + 11,\ z = -2s - 4$

***Cross Product*** **In Exercises 25 and 26, (a) find the coordinates of three points $P$, $Q$, and $R$ in the plane, and determine the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (b) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. What is the relationship between the components of the cross product and the coefficients of the equation of the plane? Why is this true?**

**25.** $4x - 3y - 6z = 6$




**26.** $2x + 3y + 4z = 4$




**In Exercises 27–32, find an equation of the plane passing through the point perpendicular to the indicated vector or line.**

| | *Point* | *Perpendicular to* |
|---|---|---|
| **27.** | $(2, 1, 2)$ | $\mathbf{n} = \mathbf{i}$ |
| **28.** | $(1, 0, -3)$ | $\mathbf{n} = \mathbf{k}$ |
| **29.** | $(3, 2, 2)$ | $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ |
| **30.** | $(0, 0, 0)$ | $\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$ |
| **31.** | $(0, 0, 6)$ | $x = 1 - t,\ y = 2 + t,\ z = 4 - 2t$ |
| **32.** | $(3, 2, 2)$ | $\dfrac{x - 1}{4} = y + 2 = \dfrac{z + 3}{-3}$ |

**In Exercises 33–44, find an equation of the plane.**

**33.** The plane passes through $(0, 0, 0)$, $(1, 2, 3)$, and $(-2, 3, 3)$.

**34.** The plane passes through $(2, 3, -2)$, $(3, 4, 2)$, and $(1, -1, 0)$.

**35.** The plane passes through $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, -2, 2)$.

**36.** The plane passes through the point $(1, 2, 3)$ and is parallel to the $yz$-plane.

**37.** The plane passes through the point $(1, 2, 3)$ and is parallel to the $xy$-plane.

**38.** The plane contains the $y$-axis and makes an angle of $\pi/6$ with the positive $x$-axis.

**39.** The plane contains the lines given by

$$\frac{x - 1}{-2} = y - 4 = z \quad \text{and} \quad \frac{x - 2}{-3} = \frac{y - 1}{4} = \frac{z - 2}{-1}$$

**40.** The plane passes through the point $(2, 2, 1)$ and contains the line given by

$$\frac{x}{2} = \frac{y - 4}{-1} = z.$$

**41.** The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

**42.** The plane passes through the points $(3, 2, 1)$ and $(3, 1, -5)$ and is perpendicular to the plane $6x + 7y + 2z = 10$.

**43.** The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the $x$-axis.

**44.** The plane passes through the points $(4, 2, 1)$ and $(-3, 5, 7)$ and is parallel to the $z$-axis.

**In Exercises 45–50, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.**

45. $5x - 3y + z = 4$
$x + 4y + 7z = 1$

46. $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

47. $x - 3y + 6z = 4$
$5x + y - z = 4$

48. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

49. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

50. $2x - z = 1$
$4x + y + 8z = 10$

**In Exercises 51–58, mark any intercepts and sketch a graph of the plane.**

51. $4x + 2y + 6z = 12$

52. $3x + 6y + 2z = 6$

53. $2x - y + 3z = 4$

54. $2x - y + z = 4$

55. $y + z = 5$

56. $x + 2y = 4$

57. $x = 5$

58. $z = 8$

**In Exercises 59–62, use a computer algebra system to graph the plane.**

59. $2x + y - z = 6$

60. $x - 3z = 3$

61. $-5x + 4y - 6z = -8$

62. $2.1x - 4.7y - z = -3$

**In Exercises 63 and 64, determine if any of the planes are parallel or identical.**

63. $P_1$: $3x - 2y + 5z = 10$
$P_2$: $-6x + 4y - 10z = 5$
$P_3$: $-3x + 2y + 5z = 8$
$P_4$: $75x - 50y + 125z = 250$

64. $P_1$: $-60x + 90y + 30z = 27$
$P_2$: $6x - 9y - 3z = 2$
$P_3$: $-20x + 30y + 10z = 9$
$P_4$: $12x - 18y + 6z = 5$

**In Exercises 65 and 66, describe the family of planes represented by the equation, where $c$ is any real number.**

65. $x + y + z = c$

66. $cy + z = 0$

**In Exercises 67 and 68, find a set of parametric equations for the line of intersection of the planes.**

67. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

68. $6x - 3y + z = 5$
$-x + y + 5z = 5$

**In Exercises 69–72, find the point(s) of intersection (if any) of the plane and the line. Also determine whether the line lies in the plane.**

69. $2x - 2y + z = 12,\quad x - \frac{1}{2} = \frac{y + (3/2)}{-1} = \frac{z + 1}{2}$

70. $2x + 3y = -5,\quad \frac{x - 1}{4} = \frac{y}{2} = \frac{z - 3}{6}$

71. $2x + 3y = 10,\quad \frac{x - 1}{3} = \frac{y + 1}{-2} = z - 3$

72. $5x + 3y = 17,\quad \frac{x - 4}{2} = \frac{y + 1}{-3} = \frac{z + 2}{5}$

**In Exercises 73–76, find the distance between the point and the plane.**

73. $(0, 0, 0)$
$2x + 3y + z = 12$

74. $(0, 0, 0)$
$8x - 4y + z = 8$

75. $(2, 8, 4)$
$2x + y + z = 5$

76. $(3, 2, 1)$
$x - y + 2z = 4$

**In Exercises 77–80, find the distance between the planes.**

77. $x - 3y + 4z = 10$
$x - 3y + 4z = 6$

78. $4x - 4y + 9z = 7$
$4x - 4y + 9z = 18$

79. $-3x + 6y + 7z = 1$
$6x - 12y - 14z = 25$

80. $2x - 4z = 4$
$2x - 4z = 10$

**In Exercises 81 and 82, find the distance between the point and the line given by the set of parametric equations.**

81. $(1, 5, -2)$; $x = 4t - 2, y = 3, z = -t + 1$

82. $(1, -2, 4)$; $x = 2t, y = t - 3, z = 2t + 2$

### Getting at the Concept

83. Give the parametric equations and the symmetric equations of a line in space. Describe what is required to find these equations.

84. Give the standard equation of a plane in space. Describe what is required to find this equation.

85. Describe a method of finding the line of intersection of two planes.

86. Describe each surface given by the equations $x = a$, $y = b$, and $z = c$.

87. (a) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are 4 units from the point $(3, -2, 5)$.
(b) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are 4 units from the plane
$4x - 3y + z = 10$.

88. Consider the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Describe the geometric figure generated by the terminal points of the following vectors, where $s$ and $t$ represent all real numbers.
(a) $t\mathbf{v}$ (b) $\mathbf{u} + t\mathbf{v}$ (c) $s\mathbf{u} + t\mathbf{v}$