Limits and Continuity

Def 20.1: Limit of f at c.
Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. We say that a real number L is a limit of f at c, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $0 < |x - c| < \delta$.

Thm 20.2 Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim\limits_{x \to c} f(x) = L$ iff for each neighborhood V of L there exists a deleted neoghborhood U of c such that $f(U \cap D) \subset V$.

Thm 20.8 Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim\limits_{x \to c} f(x) = L$ iff for every sequence $\{s_n\}$ in D that converges to c with $s_n \neq c$ $\forall n$, the sequence $\{f(s_n)\}$ converges to L.

Corollary 20.9 If $f : D \to \mathbb{R}$ and if c is an accumulation point of D, then f can have only one limit at c.

Thm 20.10 Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then TFAE:
(a) f does not have a limit at c.
(b) There exists a sequence $\{s_n\}$ in D with $s_n \neq c$ $\forall n$, such that $\{s_n\}$ converges to c, but $\{f(s_n)\}$ is not convergent in $\mathbb{R}$.

Def. 20.12

(a) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. We define the sum $f + g$ and the product fg to be the functions from D to $\mathbb{R}$ given by
$(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for all $x \in D$.
.

(b) If $k \in \mathbb{R}$, then the multiple $kf : D \to \mathbb{R}$ is the function defined by
$(kf)(x) = kf(x)$ for all $x \in D$.
.

(c) If $g(x) \neq 0$ for all $x \in D$, then the quotient $f/g : D \to \mathbb{R}$ is the function defined by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ for all $x \in D$.

Thm 20.13 Let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ and c be an accumulation point of D. if $\lim\limits_{x \to c} f(x) = L$, $\lim\limits_{x \to c} g(x) = M$, and $k \in \mathbb{R}$, then
(a) $\lim\limits_{x \to c} (f + g)(x) = L + M$
(b) $\lim\limits_{x \to c} (fg)(x) = LM$

(c) $\lim_{x\to c}(kf)(x) = kL$

(d) $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$, provided $g(x) \neq 0$ for all $x \in D$ and $M \neq 0$.

HW: 20.1, 20.3 (a,c,e,g), 20.4, 20.6 (c), 20.7 (c), 20.9 (a).