

Subsequences

Def 19.1: Subsequence

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $\{n_k\}_{k=1}^{\infty}$ be any sequence of natural numbers such that $n_1 < n_2 < n_3 < \dots$. The sequence $\{s_{n_k}\}_{k=1}^{\infty}$ is a subsequence of the sequence $\{s_n\}_{n=1}^{\infty}$.

Thm 19.4 If a sequence $\{s_n\}_{n=1}^{\infty}$ converges to a real number s , then every subsequence of $\{s_n\}_{n=1}^{\infty}$ also converges to s .

Thm 19.7 Every bounded sequence has a convergent subsequence.

Thm 19.8 Every unbounded sequence contains a monotone subsequence that has either $+\infty$ or $-\infty$ as a limit.

Def. 19.9 (a) Subsequential limit

Let $\{s_n\}$ be a bounded sequence. A subsequential limit of $\{s_n\}$ is any real number that is the limit of some subsequence of $\{s_n\}$.

Def. 19.9 (b) Limit superior (limsup)

If S is the set of all subsequential limits of $\{s_n\}$, then we define the limit superior of $\{s_n\}$ to be

$$\limsup s_n = \sup S$$

Def. 19.9 (c) Limit inferior (liminf)

If S is the set of all subsequential limits of $\{s_n\}$, then we define the limit inferior of $\{s_n\}$ to be

$$\liminf s_n = \inf S$$

Thm 19.11 Let $\{s_n\}$ be a bounded sequence and let $m = \limsup s_n$. Then the following properties hold:

(a) For every $\varepsilon > 0$ there exists a number N such that $n > N$ implies that $s_n < m + \varepsilon$.

(b) For every $\varepsilon > 0$ and for every $i \in \mathbb{N}$ there exists an integer $k > i$ such that $s_k > m - \varepsilon$.

Furthermore, if m is a real number satisfying properties (a) and (b) then $m = \limsup s_n$.

Corollary 19.12 Let $\{s_n\}$ be a bounded sequence and let $m = \limsup s_n$. Then $m \in S$, where S is the set of subsequential limit points of $\{s_n\}$. That is, there exists a subsequence of $\{s_n\}$ that converges to m .

Thm 19.14 Suppose that $\{r_n\}$ converges to a positive number r and $\{s_n\}$ is a bounded sequence. Then $\limsup r_n s_n = r \limsup s_n$.

HW: 19.1, 19.3, 19.4, 19.5, 19.9, 19.11.