Limit Theorems

Theorem 17.1
Suppose that $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then

a. $\lim_{n\to\infty} (s_n + t_n) = s + t$
b. $\lim_{n\to\infty} (ks_n) = ks$
c. $\lim_{n\to\infty} (s_n t_n) = st$
d. $\lim_{n\to\infty} (s_n/t_n) = s/t$, provided $t_n \neq 0$ for all $n \in \mathbb{N}$ and $t \neq 0$.

Theorem 17.4
Suppose that $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences with $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. If $s_n \leq t_n$ for all $n \in \mathbb{N}$ then $s \leq t$.

(*) Corollary 17.5
If $\lim_{n\to\infty} t_n = t$ and $t_n \geq 0$ for all $n \in \mathbb{N}$, then $t \geq 0$.

(*) Theorem 17.7
Suppose that $\{s_n\}_{n=1}^{\infty}$ is a sequence of positive terms and that the sequence of ratios $\{s_{n+1}/s_n\}_{n=1}^{\infty}$ converges to L. If $L < 1$ then $\lim_{n\to\infty} s_n = 0$.

Def 17.9 Convergence to infinity
A sequence $\{s_n\}_{n=1}^{\infty}$ is said to diverge to $+\infty$, and we write $\lim_{n\to\infty} s_n = +\infty$ provided that:
for every $M \in \mathbb{R}$ there exist a number N such that $n > N$ implies that $s_n > M$.
Similarly, $\{s_n\}_{n=1}^{\infty}$ is said to diverge to $-\infty$, and we write $\lim_{n\to\infty} s_n = -\infty$ provided that:
for every $M \in \mathbb{R}$ there exist a number N such that $n > N$ implies that $s_n < M$.
Very important note: Neither $\lim_{n\to\infty} s_n = +\infty$ nor $\lim_{n\to\infty} s_n = -\infty$ implies that the limit exists.
For the limit to exist the sequence has to converge to a real number.

Theorem 17.12
Suppose that $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences such that $s_n \leq t_n$for all $n \in \mathbb{N}$.
a. If $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} t_n = +\infty$.
b. If $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$.

Theorem 17.13
Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, then $\lim_{n\to\infty} s_n = +\infty$ iff $\lim_{n\to\infty} \left(\frac{1}{s_n}\right) = 0$.

HW: 17.1, 17.2, 17.4 (b), 17.5.