Topology of the reals

Def 13.1: $\varepsilon-$neighborhood of x
Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. An $\varepsilon-$neighborhood of x is a set of the form $N(x;\varepsilon) := \{y \in \mathbb{R} : |x - y| < \varepsilon\}$.

Def 13.2: Deleted neighborhood
Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A deleted neighborhood of x is a set of the form $N^*(x;\varepsilon) := \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\} = N(x;\varepsilon)\backslash\{x\}$.

Def 13.14: Limit point/accumulation point
Let $E \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ (might not be in E) is called a limit point/accumulation point of E if $E \cap N^*(x;\varepsilon) \neq \emptyset$ **for all** $\varepsilon > 0$.
The set of all limit points of E is denoted E'.

Def 13.14 Isolated point
If $x \in E \subseteq \mathbb{R}$ and $x \notin E'$ then x is an isolated point of E.

Def 13.16: Closure
Let $E \subseteq \mathbb{R}$. The closure of E is $\overline{E} := E \cup E'$.

Def 13.3 (a): Interior point
Let $E \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is an interior point of E if $\exists \varepsilon > 0$ such that $N(x;\varepsilon) \subseteq E$.
The set of all interior points of E is denoted E^o. Clearly, $E^o \subseteq E$.

Def 13.3 (b): Boundary point
Let $E \subseteq \mathbb{R}$. The boundary of E is $\partial E := \overline{E} \cap (\overline{\mathbb{R}\backslash E})$. The set of points that are arbitrarily close to E and to the complement of E.

Def 13.6 (a): Closed set
We say that E is a closed subset of $\mathbb{R}$ if $E = \overline{E}$.

Def 13.6 (b): Open set
We say that E is an open subset of $\mathbb{R}$ if $E = E^o$.

Remark
The following possibilities can occur:
1. E is open.
2. E is closed.
3. E is open and closed.
4. E is neither open nor closed.

Def. Dense
A subset E of $\mathbb{R}$ is said to be dense in $\mathbb{R}$ if $\overline{E} = \mathbb{R}$.

Thm. 13.7
(a) E is an open subset of $\mathbb{R}$ iff $E = E^o$, equivalently, E is open iff every point in E is an interior point of E.
(b) A set E is closed iff its complement $\mathbb{R}\backslash E$ is open.

Thm. 13.10, 13.11
(a) The union of any collection of open sets is open.
(b) The intersection of any finite collection of open sets is open.
(c) The intersection of any collection of closed sets is closed.
(d) The union of any finite collection of closed sets is closed.

Thm. 13.17
Let $E \subseteq \mathbb{R}$. Then
(a) E is closed iff $E' \subseteq E$,
(b) $\overline{E}$ is closed,
(c) E is closed iff $\overline{E} = E$,
(d) $\overline{E} = E \cup \partial E$.

HW: 13.1, 13.2, 13.3 (a,c), 13.4 (a,c), 13.5 (a,b,c,d), 13.6 (a,b,c,d), 13.7, 13.11, 13.19.