

The real numbers

Def 1: Order

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

- (i) If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $x > y$ is true.
- (ii) If $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.

Def 2: Ordered set

An ordered set is a set S on which an order is defined.

Def 3 (12.2): Bounded above, upper bound, maximum

Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E . If $\beta \in E$ then $\beta = \max S$.

Lower bounds and minima are defined the same way (with \geq in place of \leq).

Def 4 (12.2): Bounded set

An ordered set S is bounded if it has an upper and a lower bound.

Def 5: Least upper bound

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- (i) α is an upper bound for E .
- (ii) If $\gamma < \alpha$ then γ is not an upper bound for E .

Then α is called the least upper bound of E , or the supremum of E , and we write $\alpha = \sup E$.

The greatest lower bound, or infimum, of a set E which is bounded below is defined in the same manner. The statement $\alpha = \inf E$ means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E .

Def 6: Least-upper-bound property

An ordered set S is said to have the least-upper-bound property if the following is true:

If $E \subset S$, $E \neq \emptyset$ and E is bounded above, then $\sup E$ exists in S .

Def 7: Field

A field is a set F with two operations, called addition and multiplication, which satisfy the following so-called "field axioms" (A), (M), and (D).

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then $x + y \in F$.

(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.

- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then $xy \in F$.
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) To every $x \in F$ corresponds an element $\frac{1}{x} \in F$ such that $x\frac{1}{x} = 1$.

(D) The distributive law

$$x(y + z) = xy + xz \text{ for all } x, y, z \in F.$$

Def 8: Ordered Field

An ordered field is a field F which is also an ordered set, such that:

- (i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$.
- (ii) $xy > 0$ if $x \in F, y \in F, x > 0$ and $y > 0$.

Proposition 9: The following statements are true in every ordered field.

- (a) If $x > 0$ then $-x < 0$, and viceversa.
- (b) If $x > 0$ and $y < z$ then $xy < xz$.
- (c) If $x < 0$ and $y < z$ then $xy > xz$.
- (d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
- (e) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Theorem 10: There exists an ordered field \mathbb{R} which has the least upper bound property. Moreover, \mathbb{R} contains \mathbb{Q} as a subfield.

Theorem 11 (11.7): Let $x, y \in \mathbb{R}$ such that $x \leq y + \varepsilon$ for every $\varepsilon > 0$. Then $x \leq y$.

Def. 12 (11.8): Absolute value

If $x \in \mathbb{R}$, then the absolute value of x , denoted by $|x|$, is defined by $|x| =$

$$\begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Theorem 13 (11.9): Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

- (a) $|x| \geq 0$,
- (b) $|x| \leq a$ iff $-a \leq x \leq a$,
- (c) $|xy| = |x||y|$,
- (d) $|x + y| \leq |x| + |y|$.

Theorem 14 (12.10 and 12.12): Let $x \in \mathbb{R}, y \in \mathbb{R}$.

- (a) If $x > 0$, then there is $n \in \mathbb{N}$ such that $y < nx$. (Archimedean property of \mathbb{R}).
- (b) If $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$. (\mathbb{Q} is dense in \mathbb{R} : between any two real numbers there is a rational one).

HW: 10.3, 10.7, 10.18, 11.4, 11.6, 11.7, 11.10, 12.1, 12.2, 12.3, 12.4.