The Riemann Integral, properties and Fundamental Theorem of Calculus

Def 29.1: Let $[a, b]$ be an interval in $\mathbb{R}$. A partition P of $[a, b]$ is a finite set of points $\{x_0, x_1, ..., x_n\}$ in $[a, b]$ such that $a = x_0 < x_1 < ... < x_n = b$.

Def 29.2 Suppose that f is a bounded function defined on $[a, b]$ and that $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$.
For each $i = 1, ..., n$ we let
$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and
$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.
Letting $\Delta_{x_i} = x_{i-1}, x_i$, we define the upper sum of f with respect to P to be :

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta_{x_i},$$

and the lower sum of f with respect to P to be:

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta_{x_i},$$

Def. 29.3 Let f be a bounded function defined on an interval $[a, b]$. Then
$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$
is called the upper integral of f on $[a, b]$.
Similarly
$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$
is called the lower integral of f on $[a, b]$.
If these upper and lower integrals are equal, then we say that f is Riemann integrable on $[a, b]$, and we denote their common value by $\int_a^b f$ or $\int_a^b f(x)dx$.
That is, if $L(f) = U(f)$, then $\int_a^b f = \int_a^b f(x)dx = L(f) = U(f)$.

Thm 29.4 Let f be a bounded function defined on an interval $[a, b]$. If P and Q are partitions of $[a, b]$ and Q is a refinement of P, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.

Thm 29.6 Let f be a bounded function defined on an interval $[a, b]$. Then $L(f) \leq U(f)$.

Thm 29.9 Let f be a bounded function defined on an interval $[a, b]$, then f is integrable iff for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Thm 30.1 Let f be a monotone function defined on an interval $[a, b]$, then f is integrable on $[a, b]$.

Thm 30.2 Let f be a continuous function defined on an interval $[a, b]$, then f is integrable on $[a, b]$.

Thm 30.4 Let f and g be integrable functions on an interval $[a, b]$ and let $k \in \mathbb{R}$. Then:
(a) kf is integrable and $\int_a^b kf = k \int_a^b f$.
(b) $f + g$ is integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

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Thm 30.6 Let f be integrable on $[a, c]$ and $[c, b]$. Then f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Thm 30.7 Let f be integrable on $[a, b]$ and g be continuous on $[c, d]$. Where $f([a, b]) \subseteq [c, d]$. Then $g \circ f$ is integrable on $[a, b]$.

Fundamental Theorem of Calculus 1
If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$ then $\int_a^b f'(x)dx = f(b) - f(a)$.

Fundamental Theorem of Calculus 2
If f is continuous on an interval I containing a, then, for every x in the interval, $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.