

Differentiation and Mean value theorem for derivatives

Def 25.1: Let f be a real valued function defined on an interval I containing c . We say that f is differentiable at c if the limit $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists and is finite.

The derivative of f at c is denoted by $f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ whenever the limit exists and is finite.

Thm 25.3 Let I be an interval containing c and suppose that $f : I \rightarrow \mathbb{R}$. Then f is differentiable at c iff, for every sequence $\{x_n\}_{n=1}^{\infty}$ in I that converges to c with $x_n \neq c$ for all n , the sequence $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_{n=1}^{\infty}$ converges. Furthermore, if f is differentiable at c then the sequence of quotients above will converge to $f'(c)$.

Thm 25.6 If $f : I \rightarrow \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c .

Thm 25.7 Suppose that $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ are differentiable at $c \in I$, then:

- (a) If $k \in \mathbb{R}$, then the function kf is differentiable at c and $(kf)'(c) = k \cdot f'(c)$.
- (b) The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.
- (c) The function fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.
- (d) If $g(c) \neq 0$ then the function f/g is differentiable at c and $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$.

Thm 25.10 (Chain Rule) Let I and J be intervals in \mathbb{R} , let $f : I \rightarrow \mathbb{R}$ and $g : J \rightarrow \mathbb{R}$, where $f(I) \subseteq J$, and let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Thm 26.1 if f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Thm 26.2 (Rolle's Theorem) Let f be continuous on $[a, b]$, differentiable on (a, b) and such that $f(a) = f(b)$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Thm 26.3 (Mean Value Theorem) Let f be continuous on $[a, b]$, differentiable on (a, b) . Then there exists at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Thm 26.6 Let f be continuous on $[a, b]$, differentiable on (a, b) . If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Thm 26.8 Let f be differentiable on an interval I . Then

- (a) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I .
- (b) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I .