

Solution to Problem #1657 from the December, 2002, issue of Mathematics Magazine

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Let f_j be the j th Fibonacci number, let N denote the natural numbers, and for a natural number k , let $[k] = \{1, \dots, k\}$.

Let W be the set of all finite strings of 1's and 2's. Make one minor (and artificial) modification to the set W . If the last digit in a word is 2, add the digit 0 to the end of the word. (In what follows, if the last digit is zero, it will eventually be ignored or changed to a 1.). Thus,

$$W = \{a_1 a_2 \cdots a_k \mid a_j \in \{1, 2\} (1 \leq j \leq k-1), a_k \in \{0, 1\}\}.$$

For each $w = a_1 a_2 \cdots a_k \in W$, let $s(w) = \prod_{j=1}^k a_j f_j$. Let $S = \{s(w) \mid w \in W\}$. Clearly, $S \subseteq N$. We wish to show that $S = N$.

Let $w = a_1 a_2 \cdots a_k \in W$. We make some additional definitions:

Say an index $j \in [k]$ is relevant if $j > 1$ and $a_{j-1} = a_j = 2$.

Let $\text{Rel}(w) = \{j \in [k] \mid j \text{ is relevant}\}$.

$$\text{Let count}(w) = \prod_{j \in \text{Rel}(w)} (k - j).$$

Say w is reduced if $\text{Rel}(w) = \emptyset$. Thus, if w is reduced, $\text{count}(w) = 0$ and w has no consecutive 2's. The function count is, in some sense, a measure of how far a word is from being reduced.

Lemma: Each word in W can be reduced. That is, for each $w \in W$, there exists a reduced $\hat{w} \in W$ such that $s(\hat{w}) = s(w)$.

Proof: Use complete induction on $n = \text{count}(w)$. The base case (when $\text{count}(w) = 0$) is clear. Let $n \geq 0$ and assume that any word with count less than or equal to n can be reduced. Let $w = a_1 a_2 \cdots a_k \in W$ and suppose $\text{count}(w) = n + 1$. Since $\text{count}(w) \geq 1$, the set $\text{Rel}(w)$ is nonempty. Let m be the largest element in $\text{Rel}(w)$, and note that $k - m \geq 1$ (since $a_k \neq 2$).

We know $a_{m-1} = 2$, $a_m = 2$, and $a_{m+1} \in \{0, 1\}$. Modify w to obtain w' as follows. Let $w' = b_1 b_2 \cdots b_k \in W$, where $b_{m-1} = 1$, $b_m = 1$, $b_{m+1} = a_{m+1} + 1$, and $b_j = a_j$ for all other indices in $[k]$. Since $f_{m-1} + f_m = f_{m+1}$, we have $s(w') = s(w)$. In w' , the indices $m-1$ and m are not relevant, but the index $m+2$ (if it exists) may or may not be relevant. If it is relevant, then $\text{count}(w')$ is at least 2 less than $\text{count}(w)$. If it is not, $\text{count}(w)$ is at least $k - m$ less than $\text{count}(w)$. In either case, $\text{count}(w)$ is less than $\text{count}(w)$. By induction, w' can be reduced. \square

Theorem: $S = N$. That is, for each $n \in N$, there exists a $w \in W$ such that $s(w) = n$.

Proof: Use induction on n . The base case is clear since $1 = 1 \cdot f_1$. Suppose that $n \geq 1$ and there exists a $w = a_1 a_2 \cdots a_k \in W$ such that $s(w) = n$. By the lemma, we may assume w is reduced. Thus there is a $p \in \{1, 2\}$ such that $a_p = 1$. Modify w to obtain w' as follows. Let $w' = b_1 b_2 \cdots b_k \in W$, where $b_p = 2$ and $b_j = a_j$ for all other indices in $[k]$. Since $f_1 = f_2 = 1$, we have $s(w') = n + 1$. \square