Exploring Schur Numbers

Keith Brandt
Rockhurst University

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Don's question:

Find a 3-coloring of the integers 1, 2, ..., 13 for which the equation $x + y = z$ is never satisfied within the same color.

In theory, the decision tree will have $3^{11} = 177,147$ paths to consider.

By skipping the dead end branches (backtracking), we can search a small portion of the tree.

My Goal:

Write a computer program to find a solution.

Issai Schur:

Born 1875 in Mogilyov province, now Belarus.

Educated in Berlin.

Spent all but a few years of his career in Berlin.

Died 1941 in Tel Aviv.

Shur's mathematical ancestry is interesting:

Gauss

Gudermann

Weierstrass

Frobenius

Schur

Brauer

Schur numbers:

The Schur number S(k) is the smallest integer n for which any k-coloring of the integers 1 through n gives a solution to $x + y = z$, with x, y, and z all in the same color.

A solution to Don's question would show $S(3) > 13$.

$S(1) = 2$, since $1 + 1 = 2$.

$S(2) = 5$:

Program to compute S(3):

Variables:

Solution = string of r's, w's, and b's.

red, white, blue = strings of 0's & 1's.

redsums, whitesums, bluesums = strings of 0's & 1's.

Procedure **update** is called when a given integer can be colored a certain color:

```
procedure update(var c : color; var csum : sums;
number : integer);
  var i : integer;
  begin
  c[number]:=1; csum[2*number]:=1;
  for i:=1 to 13 do if c[i]=1 then csum[i+number]:=1;
  end;
```

Recursive procedure **try** is the heart of the program:

```
procedure try(x,k : integer; r,w,b : color; rsums,
wsums, bsums : sums);
  begin
  if x=1 then update(r,rsums,k-1);
  if x=2 then update(w,wsums,k-1);
  if x=3 then update(b,bsums,k-1);
  if k=14 then present(solution);
    else
    begin
    if rsums[k]=0 then
      begin
      solution[k]:='r';
      try(1,k+1,r,w,b,rsums,wsums,bsums);
      end;
    if wsums[k]=0 then
      begin
      solution[k]:='w';
      try(2,k+1,r,w,b,rsums,wsums,bsums);
      end;
    if bsums[k]=0 then
      begin
      solution[k]:='b';
      try(3,k+1,r,w,b,rsums,wsums,bsums);
      end;
    end;
  end;
```

Main body of program:

```
begin
initialize;
solution[1]:='r';
update(red,redsums,1);
solution[2]:='w';
try(2,3,red,white,blue,redsums,whitesums,bluesums)
end.
```

The output:

Solution 1: r w w r b b r b b r w w r

Red sums: 2 5 8 11 14 17 20 23 26
White sums: 4 5 6 13 14 15 22 23 24
Blue sums: 10 11 12 13 14 15 16 17 18

Solution 2: r w w r b b w b b r w w r

Red sums: 2 5 8 11 14 17 20 23 26
White sums: 4 5 6 9 10 13 14 15 18 19 22 23 24
Blue sums: 10 11 12 13 14 15 16 17 18

Solution 3: r w w r b b b b b r w w r

Red sums: 2 5 8 11 14 17 20 23 26
White sums: 4 5 6 13 14 15 22 23 24
Blue sums: 10 11 12 13 14 15 16 17 18

Theorem: S(3)=14.

What about S(4)??

Easy to see that S(4) > 27. We can append 14 g's to any one of the solutions above.

But now the decision tree is getting pretty large:

$4^{25} \cong 1.1 \times 10^{15}$.

Modify program and consider 4 colors.

Results:

Program found 546 4-colorings of 1 through 44 and no 4-colorings of 1 through 45. Thus,

Theorem (Baumert, 1965): S(4) = 45.

Note: $4^{43} \cong 7.7 \times 10^{25}$. Wow!

Program runs in about 30 minutes. Procedure **try** gets called over 441 million times.

What about S(5)??

Note that S(5) > 89 . We can append 45 p's to any one of the solutions above.

But now the decision tree is really getting out of hand:

$5^{87} \cong 6.5 \times 10^{60}$.

Good luck!

What else is known?

Schur (1916) showed that

- $S(k) > \dfrac{(3^k - 1)}{2}$

- $S(k) \leq R(k) - 1$, where $R(k)$ is the k^{th} Ramsey number.

Fredricksen (1973) and Exoo (1994) showed that

$$160 \leq S(5) \leq 316.$$

Fredricksen and Sweet (2000) showed that

$$S(6) \geq 536 \quad \text{and} \quad S(7) \geq 1680.$$

For more info, visit Mathworld.Wolfram.com

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Appendix:

Peter Simone's 3-coloring of 1, 2, ..., 13:

Start with a block of b's in the middle, and note that no other slots can be b's.

1	2	3	4	5	6	7	8	9	10	11	12	13
				b	b	b	b	b				