

**Building a Sine Calculator:
A Project for Calculus II Students**

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Audience:

Students in Calculus II who have studied the Lagrange error bound for Taylor series.

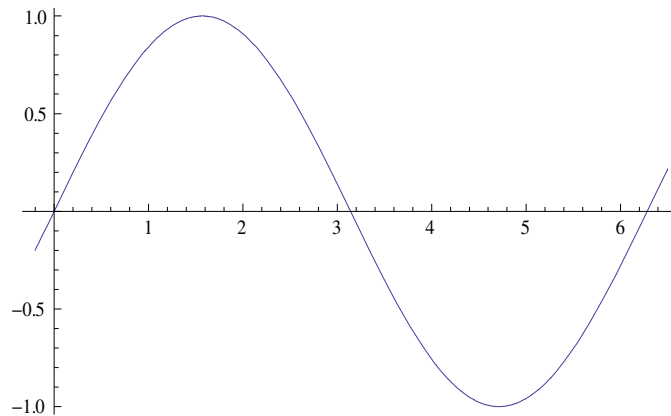
Goals:

- Write a *Mathematica* notebook that computes the sine function.
- Illustrate/review/use several key concepts from trigonometry and calculus.

We will use a Taylor series to approximate the sine function.

Basic Fact I from Calculus: A Taylor series is the most accurate near the point where we build it.

Thus, exploit the periodic nature of the sine function to consider values as close as possible to $x = 0$.



Any initial input x_1 can be "reduced" to an angle between 0 and 2π .

Let $x_2 = \text{Mod}[x_1, 2\pi]$.

We can further reduce by observing that if $\pi < x < 2\pi$, then $\sin x = -\sin(x - \pi)$. Thus by keeping track of the sign, we can reduce to an angle between 0 and π .

Let $x_3 = \text{Mod}[x_2, \pi]$

Finally, since the sine function is symmetric about the point $x = \pi/2$, we know $\sin x = \sin(\pi - x)$.

Let $x_4 = \text{Min}[x_3, \pi - x_3]$.

Now we have x_4 between 0 and $\pi/2$, and we know $\sin(x_4) = \pm \sin(x_1)$.

The Taylor series for $\sin x$ (at $x = 0$),

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \dots,$$

converges for all x .

Given a desired accuracy, how many terms of the series should we use?

Lagrange Error Bound:

If $P_n(x)$ is the n^{th} Taylor polynomial of $f(x)$ about $x = 0$, then

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1},$$

Where $|f^{(n+1)}(t)| \leq M$ for all t between 0 and x .

Where does this bound come from?

Basic Fact II from Calculus: If $f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Let $E_n(x) = f(x) - P_n(x)$.

By the way the Taylor polynomials are constructed, we know $E_n^{(k)}(0) = 0$ for $k = 0, 1, \dots, n$ and $E_n^{(n+1)}(x) = f^{(n+1)}(x)$.

For t between 0 and x ,

$$-M \leq E_n^{(n+1)}(t) \leq M,$$

So by BF II,

$$-\int_0^x Mdt \leq \int_0^x E_n^{(n+1)}(t)dt \leq \int_0^x Mdt.$$

Thus

$$-Mx \leq E_n^{(n)}(x) \leq Mx.$$

By continuing to integrate (and use BF II), we get

$$-\frac{M}{(n+1)!}x^{n+1} \leq E_n(x) \leq \frac{M}{(n+1)!}x^{n+1}.$$

We are considering x between 0 and $\pi/2$, so for our purposes, the error bound is this:

$$|E_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \leq \frac{1}{(n+1)!}\left(\frac{\pi}{2}\right)^{n+1}.$$

Now given a desired accuracy, we can determine how many terms of the Taylor series we will need to use.

Here are a few modifications and simplifications you could consider:

- Use epsilon to determine how many decimal places to display.
- Reduce to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ instead of $[0, \frac{\pi}{2}]$.
- Find a closed form expression for $p[t]$ that doesn't use the sine function.
- To avoid some of the reducing, simply work with x between 0 and 2π .
- To avoid the while loop, students could use a table of values for $\frac{1}{(n+1)!} \left(\frac{\pi}{2}\right)^{n+1}$ to determine a suitable value for n .

Other projects to consider:

- Build a cosine calculator (boring!).
- Build a square root calculator.
- Build a natural log calculator:

My View:

The process of building such calculators strengthens our students' understanding of mathematics.